Queen's Economics Department Working Paper No. 63

THE POSSIBILITY OF OVERSUPPLY WITH EXTERNAL ECONOMIES

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10-1971

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DISCUSSION PAPER NO. 63

October, 1971

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Introduction

Until fairly recently, the Pigovian position that activities generating unpriced external economies would be undersupplied was universally accepted. It rested on the simple assertion that marginal social benefits exceeded marginal private benefits of the activity, and that, by equating the latter to the price, the marginal social benefit would exceed the price and the activity should be expanded. In 1963, Buchanan and Kafoglis (3) constructed a counterexample involving immunizations in which oversupply under independent adjustment was possible both in the case of unilateral external economies and that of reciprocal external economies. In the former case, they found that "more-than-perfect substitutability" had to exist in the objective function of the affected party. However, they derived no preconditions for oversupply to occur in the reciprocal case. Two subsequent papers attempted to explain further this paradoxical outcome. Baumol (1) implied that the Buchanan and Kafoglis (hereafter B-K) paradox could be

* The following have given excessively of their time in reading and discussing a first draft of this paper: D.A. Vardy and R.G. Lipsey of Queen's University, and J.M. Treddenick of the Royal Military College of Canada.

partly explained by a violation of the second-order conditions.[1] In fact, he makes the strong statement that "the very presence of externalities makes it less likely that the second-order conditions will be satisfied" ([1] p. 362). More recently, Vincent (6) has attempted to prove that a necessary conditon for a lower total use at the optimum is that "economies of scale in the production of benefits" on the part of one of the parties exists. This paper will show that Baumol's results are, at best, misleading, and that Vincent's alleged proof is wrong for several reasons. A correct analysis will be developed which will show the conditions required for the B-K paradox as well as other general cases.

At the outset, it is necessary to correct an unfortunate terminological error in both the Baumol and Vincent contributions - an error which has resulted in substantive mistakes. If ${}_AW$ is the objective function of individual A, and if a and b are the activity levels of individuals A and B, Baumol and Vincent would argue that B exerts a "total type of external economy" ([6], p. 977) on A if ${}_AW_b > 0$ (where subscripts refer to partial derivatives). A marginal externality is said to exist by then if ${}_AW_{ab} > 0$ (in this case, an economy). These definitions are in contradiction to normal usage.[2] Following Buchanan and Stubblebine (3), a marginal external economy exists if ${}_AW_b > 0$. The corresponding non-marginal but total

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1 Baumol is careful not to suggest that this explains the B-K paradox fully. In fn. 9, he points out that some of the B-K cases can be explained by increased efficiency in the use of inputs by central management. On this point see also Olson and Zeckhauser (5).

2 They would imply, for example, that separable externalities as treated by Davis and Whinston (4) are not externalities at all.

externality exists when ${}_AW_b = 0$, but $\int_0^b {}_AW_b\,db > 0$.

The second partial derivative, ${}_AW_{ab}$, refers to the substitutability of b for a in A's objective function. Thus, if ${}_AW_{ab} < 0$, a and b are substitutes; if ${}_AW_{ab} > 0$, they are complements.[3] Otherwise, they are unrelated (or separable). Furthermore, the existence of marginal external economies is quite compatible with ${}_AW_{ab} \gtreqless 0$. Thus, it is very misleading to suggest that ${}_AW_{ab} > 0$ implies a marginal external economy, for it is possible that b reduces the value of ${}_AW({}_AW_b < 0)$.

It is in this sense that Baumol's results are misleading. When he suggests that the very presence of externalities, if strong enough, can cause a failure of second-order conditions, what he really means is that the existence of strong substitute (or complement) relationships amongst externalities and own goods can cause a failure of second-order conditions. This we already know from purely private goods economies. The problem of the second-order conditions will be returned to later in the paper. Now, the pitfalls of Vincent's paper will be shown.

Vincent's Immunization Model

Following B-K, Vincent constructs a model in which immunizations taken by one individual (or group of individuals) imposes an external economy on another individual (or group) and vice versa. It is worthwhile reworking his reasoning since it is faulty in many ways.

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[3] Intuitively, this is fairly easy to explain. For individual A, the demand curve for a will be the curve ${}_AW_a$ versus a. An increase in b which shifts the demand curve for a to the left so that a and b are substitutes implies that ${}_AW_{ab} < 0$. Thus the latter inequality summarizes the notion of substitutability. On the other hand, if b is a complement of a, b increases will shift the demand curve right so that ${}_AW_{ab} > 0$.

Using Vincent's terminology, we write the total cost functions for the two individuals, A and B, as follows:

$$ {}_{A}L = pa + {}_{A}C(a,b) $$
$$ {}_{B}L = pb + {}_{B}C(a,b) $$

where p is the fixed price of an immunization used by A and B, and ${}_{A}C$ and ${}_{B}C$ are the expected values of costs due to illness at immunization levels a and b. The cost functions, L, are to be minimized. The properties of C, using the corrected definition of externality, are assumed to be as follows:

${}_{A}C_{a}$, ${}_{B}C_{b}$ both < 0

${}_{A}C_{b}$, ${}_{B}C_{a}$ < 0 by definition of marginal external economies

${}_{A}C_{ab}$, ${}_{B}C_{ba}$ > 0 if substitutes
< 0 if complements

${}_{A}C_{aa}$, ${}_{B}C_{bb}$ > 0 if diminishing returns to immunizations exist
< 0 if increasing returns to immunizations exist.

Vincent, by wrongly assuming that ${}_{A}C_{ab} < 0$ means that external economies exist, imposes an unnecessary restriction on his results. He thus finds that a necessary condition for the B-K paradox is that ${}_{A}C_{aa} < 0$. The restriction ${}_{A}C_{ab} < 0$ is particularly inappropriate in this example for it implies complementarity between a and b - a condition hardly coinciding with the intent of the immunization example. Even though we shall show later that the logic of his method is incorrect, it is instructive to apply his methodology without the restrictions he imposes on ${}_{A}C_{ab}$. The necessary conditions that he ought to have arrived at are much less restrictive than those actually arrived at.

Assuming that (a_{in}, b_{in}) and (a_{jt}, b_{jt}) are the outcomes of independent and joint cost minimization respectively, the first-order conditions (assuming away problems of strategy) are

i) Independent cost minimization:

$$p + {}_AC_a(a_{in}, b_{in}) = 0 \tag{1}$$

$$p + {}_BC_b(a_{in}, b_{in}) = 0 \tag{2}$$

ii) Joint cost minimization:

$$p + {}_AC_a(a_{jt}, b_{jt}) + {}_BC_a(a_{jt}, b_{jt}) = 0 \tag{3}$$

$$p + {}_BC_b(a_{jt}, b_{jt}) + {}_AC_b(a_{jt}, b_{jt}) = 0 \tag{4}$$

Vincent's methodology involves asking what conditions are necessary to attain the above first-order conditions in such a way that $a_{jt} < a_{in}$ and $b_{jt} > b_{in}$. He then considers only the first-order conditions for an increase in a. Setting aside, for the moment, the shortcomings of this approach, we can follow his method to expose some of the fallacies in his reasoning.

The only restriction on the first-order conditions for an increase in a is that ${}_BC_a < 0$. That is, a imposes a marginal external economy on individual B. This implies that (from (1) and (3)),

$${}_AC_a(a_{in}, b_{in}) < {}_AC_a(a_{jt}, b_{jt}) \tag{5}$$

Vincent then says that ${}_AC_{ab} < 0$, so that

$${}_AC_a(a_{in}, b_{jt}) < {}_AC_a(a_{in}, b_{in}) \tag{6}$$

Thus, in order that (5) hold, it is necessary that

$$ {}_A C_a(a_{jt}, b_{jt}) > {}_A C_a(a_{in}, b_{jt}) \qquad (7) $$

which implies that ${}_A C_{aa} < 0$ for some range between a_{jt} and a_{in}. Vincent then concludes that economies of scale must exist in individual A's use of a for a to fall and b to rise.

However, as mentioned above, ${}_A C_{ab} > 0$ is the assumption more in keeping with the immunization example, since this implies that a and b are substitutes in A's cost function. Let us assume that ${}_A C_{ab} > 0$ and ${}_A C_{aa} > 0$ (a and b are substitutes and no economies of scale exist). Then, from ${}_A C_{ab} > 0$,

$$ {}_A C_a(a_{in}, b_{jt}) > {}_A C_a(a_{in}, b_{in}) \qquad (8) $$

and from ${}_A C_{aa} > 0$,

$$ {}_A C_a(a_{jt}, b_{jt}) < {}_A C_a(a_{in}, b_{jt}) \qquad (9) $$

It can be seen that these results are also compatible with the required inequality above. This proves that economies of scale are not necessary for the Buchanan-Kafoglis results to occur.

In fact, we could apply the above test to all four possible combinations of ${}_A C_{aa} \gtrless 0$ and ${}_A C_{ab} \gtrless 0$ to see which assumptions are compatible with the Buchanan-Kafoglis paradox. Table I illustrates the possibilities. It is seen there that the necessary conditions for the paradox are that either a and b are substitutes in A's cost function, or a exhibits increasing returns to scale. A sufficient condition is that both these necessary conditions hold. If

Table I

Restrictions on Individual A

Scale Effects		Substitutability: $_AC_{ab} > 0$ (Substitutes)	Substitutability: $_AC_{ab} < 0$ (Complements)
	$_AC_{aa} > 0$ (Diminishing Returns)	Compatible with B-K	Not Compatible with B-K
	$_AC_{aa} < 0$ (Increasing Returns)	Compatible with B-K and sufficient	(Vincent case) Compatible with B-K

Table II

Restrictions on Individual B

Scale Effects		Substitutability: $_BC_{ba} > 0$ (Substitutes)	Substitutability: $_BC_{ba} < 0$ (Complements)
	$_BC_{bb} > 0$ (Diminishing Returns)	Compatible with B-K	Compatible with B-K and sufficient
	$_BC_{bb} < 0$ (Increasing Returns)	Not Compatible with B-K	Compatible with B-K

Vincent's method were correct, these would still not be adequate results. They are derived solely from considering the first-order conditions on A's immunization. A similar table must also be drawn up for individual B by proceeding from the first-order conditions on b under the two types of cost minimization and making different assumptions about ${}_{B}C_{ba} \gtrless 0$ and ${}_{B}C_{bb} \gtrless 0$. Table II illustrates the conditions which are compatible with the result $a_{jt} < a_{in}$ and $b_{jt} > b_{in}$. These conditions should be as important as those on A, although Vincent does not consider them.

<u>Extension to Reciprocal External Economies of Consumption</u>

The external effects arising from immunizations fall into the general category of utility-affecting externalities although both Buchanan-Kafoglis and Vincent implicitly assume that cost reduction bears a one-to-one relationship to utility. It is rather peculiar that Vincent should state that "the Buchanan-Kafoglis proposition with respect to inputs appears to apply only to inputs in a production process and cannot be extended to the case of final consumption goods that are arguments to the utility functions" ([6] p. 981). Perhaps this inference is drawn from the supposed necessity of increasing returns to a in the affected function, a condition which is usually assumed away in utility functions. In this section, it will be shown that the Buchanan-Kafoglis case can be extended to utility externalities in a straightforward manner. The objective functions become utility functions to be maximized rather than cost functions to be minimized. There is a purely heuristic reason for undertaking this extension. Since utility maximization is easier to handle intuitively, and is more general than cost minimization, it will be used rather than the latter in what follows.

The two utility functions may be written:

$$_AU = {}_AU(a,b) + {}_AU(x^A, y^A, \ldots), \quad {}_AU_a > 0, \quad {}_AU_{aa} < 0, \quad {}_AU_b > 0$$
$$_BU = {}_BU(a,b) + {}_BU(x^B, y^B, \ldots), \quad {}_BU_b > 0, \quad {}_BU_{bb} < 0, \quad {}_BU_a > 0.$$

x and y are purely private goods. The functions are written in the separable form for simplicity of results. Notice that we are assuming away increasing returns in the utility functions from the outset. Assuming as before a constant price, p, for both a and b, the first-order conditions derived for given budget constraints on A and B are:

i) Independent maximization

$$p = {}_AU_a(a_{in}, b_{in}) \qquad (10)$$
$$p = {}_BU_b(a_{in}, b_{in}) \qquad (11)$$

where ${}_AU_a$ and ${}_BU_b$ are assumed measured in some common numeraire good, so are marginal rates of substitution

ii) Joint maximization

$$p = {}_AU_a(a_{jt}, b_{jt}) + {}_BU_a(a_{jt}, b_{jt}) \qquad (12)$$
$$p = {}_BU_b(a_{jt}, b_{jt}) + {}_AU_b(a_{jt}, b_{jt}) \qquad (13)$$

Vincent's method may now be applied to this case. For individual A, since ${}_BU_a > 0$ by assumption (marginal external economies), this entails (using (10) and (12))

$${}_AU_a(a_{in}, b_{in}) > {}_AU_a(a_{jt}, b_{jt}) \qquad (14)$$

Since ${}_AU_{aa} < 0$ and $a_{jt} < a_{in}$, this means

$$ {}_AU_a(a_{in},b_{jt}) < {}_AU_a(a_{jt},b_{jt}) \tag{15}$$

In order for inequality (14) to hold, it is necessary that

$$ {}_AU_a(a_{in},b_{jt}) < {}_AU_a(a_{in},b_{in}) \tag{16}$$

Given that $b_{jt} > b_{in}$, this implies that ${}_AU_{ab} < 0$. In words, b must be a substitute for a in A's utility function.[4] That would be the necessary condition for A to consume less at the optimum, given that B were consuming more.

For individual B, as above using (11) and (13) we require that

$$ {}_BU_b(a_{in},b_{in}) > {}_BU_b(a_{jt},b_{jt}) \tag{17}$$

Now, with $b_{jt} > b_{in}$ and ${}_BU_{bb} < 0$,

$$ {}_BU_b(a_{jt},b_{in}) > {}_BU_b(a_{jt},b_{jt}) \tag{18}$$

In order to attain inequality (17), it is admissible that

$$ {}_BU_b(a_{jt},b_{in}) \gtrless {}_BU_b(a_{in},b_{in}) \tag{19}$$

This means that ${}_BU_{ba} \lesseqgtr 0$. No restrictions are placed on the substitutability of a for b in B's utility function. A sufficient condition for (17) to hold can be seen to be ${}_BU_{ba} > 0$.

In summary, using Vincent's method, we find that the only necessary conditon for the output of a to fall and b to rise is that b be a substitute for a in the utility function for A. This condition is not sufficient. A sufficient condition

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4 David Vardy has made the point that it is gross substitutability and complementarity rather than net that is being used in this paper. That is, income effects are included, and all demand curves are uncompensated. We could assume zero income effects so that the two are identical, but it does not seem to be required.

was shown to be that ${}_BU_{ba} > 0$. That is, a is complementary to b in B's utility function. This condition is not necessary, and seems unlikely to hold if the above necessary one does. We are thus left with criteria which are not overly precise. The following sections will apply difference equations to show that the Vincent method employs faulty logic. However, even a correction of the logic will not suffice, for the methodological error goes deeper than that. The correct analysis will be given in the next but one section.

<u>Difference Equation Method</u>

We shall use the same example as in the previous section. From the first-order conditions and the assumptions made, we require as before,

$$ {}_AU_a(a_{in}, b_{in}) > {}_AU_a(a_{jt}, b_{jt}) \qquad (14)$$

and

$$ {}_BU_b(a_{in}, b_{in}) > {}_BU_b(a_{jt}, b_{jt}) \qquad (17)$$

From the first-order conditions on a change in a, (10) and (12), we have

$$ {}_AU_a(a_{in}, b_{in}) - {}_AU_a(a_{jt}, b_{jt}) = {}_BU_a(a_{jt}, b_{jt})$$

$$\text{or} \quad \Delta {}_AU_a = -{}_BU_a \qquad (20)$$

Taking the total differential of ${}_AU_a$ when a and b vary, we obtain

$$ d{}_AU_a = \frac{\partial {}_AU_a}{\partial a} da + \frac{\partial {}_AU_a}{\partial b} db$$

$$ = {}_AU_{aa}\, da + {}_AU_{ab}\, db \qquad (21)$$

To simplify the mathematics, let us assume ${}_AU_{aa}$, ${}_AU_{ab}$, ${}_BU_{bb}$, and ${}_BU_{ba}$ are constant over the range involved (i.e. the demand curves of each individual for a and b respectively are linear and the substitutability of a and b is constant). We can write the above differential equation in difference form,

$$\Delta_A U_a \;=\; {}_A U_{aa}\, \Delta a \;+\; {}_A U_{ab}\, \Delta b \;=\; -{}_B U_a \qquad (22)$$

Showing this equation on a diagram is easy. Figure 1 shows A's uncompensated demand curves for a at the two equilibrium values for b. The diagram is drawn on the assumption that a falls and b rises, and that a and b are substitutes in A's demand. The distance st = ${}_B U_a$. The distance xy = $- \Delta_A U_a$. They are equal at equilibrium. From the diagram, it can be seen that whether $a_{jt} \gtrless a_{in}$ depends upon two things - the size of ${}_B U_a$ (the external effect), and the amount of the shift in the demand function (substitutability effect).

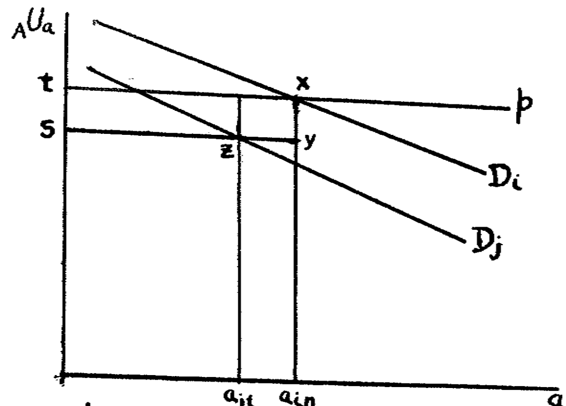


Fig. 1

Solving the above difference equation for Δa, we obtain,

$$\Delta a \;=\; \frac{-{}_B U_a}{{}_A U_{aa}} \;-\; \frac{{}_A U_{ab}}{{}_A U_{aa}}\, \Delta b \qquad (23)$$

It immediately follows that ${}_A U_{ab} < 0$ is a necessary condition for $\Delta a < 0$, given that $\Delta b > 0$ and ${}_A U_{aa} < 0$. This is consistent with earlier results. In terms of the diagram, the demand curve must shift left. Furthermore, for $\Delta a < 0$, we require that

$$-{}_{B}U_{a} - {}_{A}U_{ab}\,\Delta b < 0 \quad (\text{since } {}_{A}U_{aa} < 0)$$

That is, $-{}_{A}U_{ab}\,\Delta b > {}_{B}U_{a}$. This result can be interpreted diagrammatically as meaning that the fall in ${}_{A}U_{a}$ vertically at the old allocation (xz) must exceed the external economy from a (ts). This case is shown in Figure 1.

Turning to the individual B, we can derive an analogous difference equation from the first-order conditions on b, (11) and (13),

$$\begin{aligned} {}_{B}U_{b}(a_{in},b_{in}) - {}_{B}U_{b}(a_{jt},b_{jt}) &= {}_{A}U_{b}(a_{jt},b_{jt}) \qquad (24) \\ \Delta_{B}U_{b} &= -{}_{A}U_{b} \end{aligned}$$

From the total differential of ${}_{B}U_{b}$ we get

$$\Delta_{B}U_{b} = {}_{B}U_{bb}\,\Delta b + {}_{B}U_{ba}\,\Delta a = -{}_{A}U_{b} \qquad (25)$$

Solving for Δb,

$$\Delta b = -\frac{{}_{A}U_{b}}{{}_{B}U_{bb}} - \frac{{}_{B}U_{ba}}{{}_{B}U_{bb}}\,\Delta a \qquad (26)$$

Given that $\Delta a < 0$ and ${}_{B}U_{bb} < 0$, we can see that a sufficient condition for b to rise when a falls is ${}_{B}U_{ba} < 0$. We can also say that $\Delta b > 0$ if ${}_{A}U_{b} > {}_{B}U_{ba}\,\Delta a$. A graphical interpretation could be given as before. Notice that this is exactly the opposite sufficient condition to what we had before. Using Vincent's method, we found that a had to be a complement for b in B's function. We can now show that the Vincent method result is wrong, and that therefore his logic must be wrong.

The simplest demonstration of this fact can be given diagrammatically

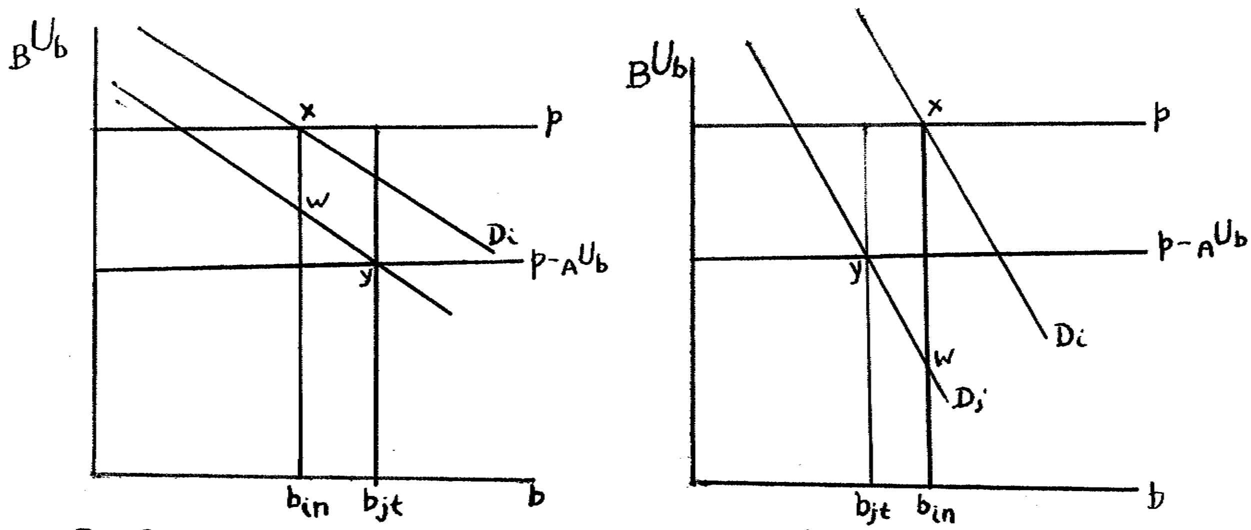


Fig 2a   Fig. 2b

The graphs show B's demand curves for b at the independent and joint equilibria. They are drawn on the assumption that $a_{jt} < a_{in}$, and that a and b are complements (the sufficient condition by Vincent's method). Thus, a fall in a shifts B's demand curve leftward.

A comparison of Figs. 2a and 2b immediately shows that a complementarity relation between a and b can cause b to rise or fall when a falls. It depends on the relative magnitudes of the external effect and the complementarity (shift in D curve). This disproves the sufficiency of a and b being complements. It can be seen that a sufficient condition is that a and b are substitutes. If the demand curve shifts right, b must increase. Thus, the difference equation method gives the correct results.

To show the error in Vincent's logical method, we can translate the proof of sufficiency derived from his method into diagrammatic terms using Fig. 2a. From the first-order conditions (11) and (13) we require

$${}_{B}U_b(a_{in}, b_{in}) > {}_{B}U_b(a_{jt}, b_{jt}) \text{ or } x > y \text{ (measured vertically)} \quad (17)$$

With $b_{jt} > b_{in}$ and ${}_{B}U_{bb} < 0$,

$${}_{B}U_b(a_{jt}, b_{in}) > {}_{B}U_b(a_{jt}, b_{jt}) \text{ or } w > y \quad (18)$$

Therefore, sufficiency requires that

$${}_{B}U_b(a_{jt}, b_{in}) < {}_{B}U_b(a_{in}, b_{in}) \text{ or } x > w \quad (19a)$$

The latter implies ${}_{B}U_{ba} > 0$ and the two must be complements. The reason this proof does not prove sufficiency is that it excludes the possibility of the case of Fig. 2b in which $y > w$ and $x > w$ so $x \gtrless y$. This occurs because one of the premises, $w > y$, presupposes the conclusion that $b_{jt} > b_{in}$. Thus, on purely logical grounds, Vincent's methodology is incorrect.

The difference equation method corrects this logical shortfall implicit in Vincent's method. It tells us the following. Given that Δb is positive, a necessary condition for a to fall is that a and b be substitutes in A's utility function; and, given that Δa is negative, a sufficient condition for b to rise is that a and b be substitutes in B's utility function. Unfortunately, these results are still not adequate because the analysis is partial. We have put preconditions on the sign of one in order to determine the sign of the other. In fact, both Δa and Δb are determined simultaneously (from a comparative static point of view). The final section takes this into account.

Simultaneous Determination of Δa and Δb

The simultaneously determined solutions for Δa and Δb are easily arrived at. Above, we have two difference equations in two unknowns;

$$\Delta a = \frac{-{}_BU_a}{{}_AU_{aa}} - \frac{{}_AU_{ab}}{{}_AU_{aa}} \Delta b \qquad (23)$$

$$\Delta b = \frac{-{}_AU_b}{{}_BU_{bb}} - \frac{{}_BU_{ba}}{{}_BU_{bb}} \Delta a \qquad (26)$$

Solving these for Δa and Δb, we obtain,

$$\Delta a = \frac{{}_AU_b\,{}_AU_{ab} - {}_BU_a\,{}_BU_{bb}}{{}_AU_{aa}\,{}_BU_{bb} - {}_AU_{ab}\,{}_BU_{ba}} \qquad (27)$$

$$\Delta b = \frac{{}_BU_a\,{}_BU_{ba} - {}_AU_b\,{}_AU_{bb}}{{}_AU_{aa}\,{}_BU_{bb} - {}_AU_{ab}\,{}_BU_{ba}} \qquad (28)$$

Equations (27) and (28) are perfectly general and can be used to tell us all we want to know about Δa and Δb. Because of the complexity of the expression, meaningful results are not completely straightforward. We proceed by the following steps - simple inspection for a priori conditions on a and b; consideration of the conditions under which $\Delta a + \Delta b < 0$; and, a consideration of the conditions under which a and b will both fall. Throughout we assume normally-shaped preferences so that ${}_AU_{aa} < 0$ and ${}_BU_{bb} < 0$. We also assume reciprocal marginal external economies so that ${}_AU_b > 0$ and ${}_BU_a > 0$.

i) Simple inspection

By inspection of equations (27) and (28), bearing in mind the above assumptions, the following can be said regarding the signs of Δa and Δb:

(a) If both a and b are complements in the other's utility function, so that ${}_BU_{ba} > 0$ and ${}_AU_{ab} > 0$, both a and b will change in the same direction. That direction will be determined by the sign of the denominator (which is the same for both). The stronger is the complementarity, the more likely is that direction to be negative. An example of reciprocal external economies which are complementary might be education. Thus, it is possible that too much education be consumed under independent adjustment. The B-K paradox applies to complements as well.

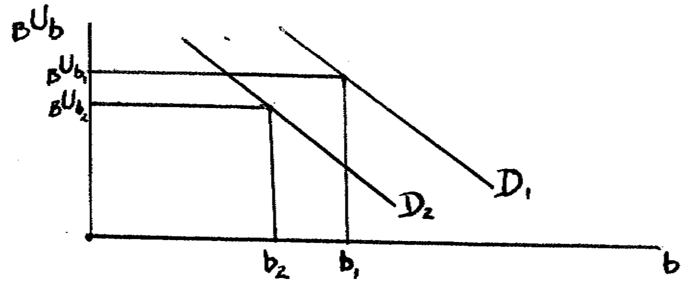
(b) If at least one of a or b is a substitute, it is no longer possible to predict anything about the signs of Δa and Δb in the absence of further knowledge about the size of the terms. The necessary and sufficient conditions derived from the partial analysis are thus invalid.

(c) In general, we can make no simple statements about the direction of influence of any one term along. No one element has an unambiguous influence - all depends on the magnitudes and signs of the other terms.

ii) <u>Conditions under which $\Delta a + \Delta b < 0$</u>

It is instructive to consider the circumstances under which total consumption will fall, not only because this is another of the Buchanan-Kafoglis paradoxes, but also because some interesting results in their own right are found.

Before proceeding, consider the notion of more-than-perfect substitutability. a will be a more-than-perfect substitute for b if ${}_BU_{ba} < {}_BU_{bb} < 0$. The easiest interpretation of this is given graphically. Figure 3 illustrates this. A unit increase in a shifts B's demand curve to the left so far (D_1 to D_2) that a unit decrease in b still leaves B with a net fall in U.



A similar expression holds for b as a more-than-perfect substitute for a – ${}_AU_{ab} < {}_AU_{aa} < 0$. The concept might also be extended to include complements. a is a more-than-perfect complement of b if $|{}_BU_{ba}| > |{}_BU_{bb}|$; and vice-versa. A similar graphical interpretation to the above could be given. A unit rise in a accompanied by a unit rise in b causes a net increase in ${}_BU_b$. The notion of more-than-perfect complementarity of external effects may be hard to imagine conceptually. One might think of the case of "keeping ahead of the Jones'" by a certain proportion as an example.

Total consumption will fall if $\Delta a + \Delta b < 0$. That is, from (1) and (2),

$$\frac{{}_AU_{b}\,{}_AU_{ab} + {}_BU_{a}\,{}_BU_{ba}}{{}_AU_{aa}\,{}_BU_{bb} - {}_AU_{ab}\,{}_BU_{ba}} < \frac{{}_AU_{b}\,{}_AU_{aa} + {}_BU_{a}\,{}_BU_{bb}}{{}_AU_{aa}\,{}_BU_{bb} - {}_AU_{ab}\,{}_BU_{ba}} \qquad (29)$$

Inspection of these equations confirms the following results:
(a) If both a and b are more-than-perfect complements, so $_A U_{ab} > |_A U_{aa}|$ and $_B U_{ba} > |_B U_{bb}|$, total consumption must fall. Furthermore, at least one of them must be a more-than-perfect complement in order for total consumption to fall. Combining this with result (a) in the previous section, we see that if total consumption of a and b falls when both are complements (at least one of which must be more-than-perfect), the consumption of both a and b must fall. If more-than-perfect complementarity is considered unlikely, then normally complementarity would cause under-supply.
(b) If both a and b are more-than-perfect substitutes, so $_A U_{ab} < {_A U_{aa}}$ and $_B U_{ba} < {_B U_{bb}}$, the denominator of both sides of the above inequality will be negative, and the inequality will not be satisfied. That is, total output will rise in the movement from the independent to the joint equilibria.
(c) If neither a nor b are more-than-perfect complements, total consumption will fall only if either a or b is a more-than-perfect substitute. This can be seen from close inspection of the above inequalities.

To summarize this section, we can say that total consumption of a and b together may fall if a and b are more-than-perfect complements (sufficient condition), if a or b is a more-than-perfect complements (not sufficient), or if a or b is a more-than-perfect substitute (not sufficient). If both a and b are more-than-perfect substitutes, total output will rise (although one may rise and the other may fall).

iii) <u>Conditions under which $\Delta a < 0$ and $\Delta b < 0$</u>

We have already shown that a and b will both fall if both are more-than-perfect complements, and may fall if at least one is a more-than-perfect complement (and the other a less-than-perfect complement). Consider now the case in which both are substitutes. It can be proven that if both are substitutes, a and b cannot both fall.

<u>Proof</u>: Suppose the denominators of (1) and (2) are negative so that ${}_AU_{ab} \cdot {}_BU_{ba} > {}_AU_{aa} \cdot {}_BU_{bb}$. From (1) and (2),

$$\Delta a < 0 \text{ if } |{}_BU_{bA}U_{ab}| < |{}_BU_{aB}U_{bb}|$$

and

$$\Delta b < 0 \text{ if } |{}_BU_{bB}U_{ba}| < |{}_AU_{bA}U_{aa}| .$$

Rearranging, these inequalities become

$$\frac{{}_AU_b}{{}_BU_a} < \left|\frac{{}_BU_{bb}}{{}_AU_{ab}}\right| \quad \text{and} \quad \frac{{}_AU_b}{{}_BU_a} > \left|\frac{{}_BU_{ba}}{{}_AU_{aa}}\right|$$

$$\therefore |{}_BU_{bbA}U_{aa}| > |{}_BU_{baA}U_{ab}|$$

But this latter contradicts the initial inequality required for the denominator to be negative so that it is not possible for a and b to fall when the denominator is negative.

An exactly analogous proof holds for the denominator being positive.

QED.

Likewise, it can be proven that both a and b cannot fall (although one may) when one is a substitute and the other a complement. In this case, the denominators in (1) and (2) are positive. The good which is a complement must rise.

The substitute may rise or fall, being more likely to fall the more substitutable it is.

One could go on and consider any number of different cases, but the more important results seem to be summarized in the above sub-sections. Oversupply by any one individual can occur in a wide variety of circumstances. Oversupply in total can occur only if one or both is a more-than-perfect complement or if one is a more-than-perfect substitute, but not both. Oversupply by each can only occur if both are complements, at least one of which is more-than-perfect.

Finally, it might be noted that the above equations apply equally as well to the case in which the external effect is only unilateral, not reciprocal. Thus, suppose B emits an externality and A does not, ${}_{A}U_{b} > 0$, ${}_{B}U_{a} = 0$. B could be poor people taking immunizations and A rich people (the Buchanan-Kafoglis case). Then, (1) and (2) become

$$\Delta a = \frac{{}_{A}U_{b}\,{}_{A}U_{ab}}{{}_{A}U_{aa}\,{}_{B}U_{bb}}$$

$$\Delta b = -\frac{{}_{A}U_{b}\,{}_{A}U_{aa}}{{}_{A}U_{aa}\,{}_{B}U_{bb}}$$

Δb is always positive, and Δa is negative if a and b are substitutes (${}_{A}U_{ab} < 0$). For total output to fall, we require that $\Delta a + \Delta b < 0$. This means that ${}_{A}U_{b}\,{}_{A}U_{ab} < {}_{A}U_{b}\,{}_{A}U_{aa}$ or ${}_{A}U_{ab} < {}_{A}U_{aa}$. That is, b must be a more-than-perfect substitute for a in A's utility function. This is exactly the same result as arrived at by Buchanan and Kafoglis.

The Second-Order Conditions

Baumol's note (1) shows that, for the simple maximization problem he sets up, if the cross-partials of the objective function (with respect to different individuals' activities) are large enough, and not of the opposite signs, the second-order conditions will be violated. In our terminology, this means that the presence of large enough substitutability or complementarity would cause failure of the second-order conditions. The previous results would be invalidated since the first-order conditions derived would no longer indicate the optimum. We ought, therefore, to consider whether or not the existence of more-than-perfect substitutability or complementarity (one of which is required for the B-K paradox) is likely to violate the second-order conditons.

The relevant second-order conditons are obtained from the maximization of A's utility subject to B's utility and a fixed total income constraint, $\bar{M}$. To simplify the problem, we assume only one private good, c, in addition to a and b. The price of c is fixed at q. The Lagrangian becomes:

$$Z = {}_AU(a, b, c_A) + \lambda_1 [{}_BU(a, b, c_B) - {}_B\bar{U}] + \lambda_2 [\bar{M} - pa - pb - q(c_A + c_B)]$$

The first-order conditions are:

$$\begin{aligned}
Z_a &= {}_AU_a + \lambda_{1B}U_a - \lambda_2 p = 0 \\
Z_b &= {}_AU_b + \lambda_{1B}U_b - \lambda_2 p = 0 \\
Z_{c_A} &= {}_AU_{c_A} - \lambda_2 q = 0 \\
Z_{c_B} &= \lambda_{1B}U_{c_B} - \lambda_2 q = 0
\end{aligned}$$

The second-order conditions in this 4-variable, 2 constraint problem become:

$$\begin{vmatrix} 0 & 0 & p & p & q \\ 0 & 0 & {}_BU_a & {}_BU_b & 0 \\ p & {}_BU_a & Z_{aa} & Z_{ab} & Z_{ac_A} \\ p & {}_BU_b & Z_{ba} & Z_{bb} & Z_{bc_B} \\ q & 0 & Z_{c_Aa} & Z_{c_Ab} & Z_{c_Ac_A} \end{vmatrix} < 0, \quad \begin{vmatrix} 0 & 0 & p & p & q & q \\ 0 & 0 & {}_BU_a & {}_BU_b & 0 & {}_BU_{c_B} \\ p & {}_BU_a & Z_{aa} & Z_{ab} & Z_{ac_A} & Z_{ac_B} \\ p & {}_BU_b & Z_{ba} & Z_{bb} & Z_{bc_A} & Z_{bcB} \\ q & 0 & Z_{c_Aa} & Z_{c_Ab} & Z_{c_Ac_A} & 0 \\ q & {}_BU_{c_B} & Z_{c_Ba} & Z_{c_Bb} & 0 & Z_{c_Bc_B} \end{vmatrix} > 0.$$

To keep the problem simple, we assume that c is separable from a and b in both utility functions. This means that Z_{ac_A}, Z_{ac_B}, Z_{bc_A}, Z_{bc_B} all vanish. Note also that $Z_{ij} = Z_{ji}$. Evaluating the determinants, the first second-order condition becomes:

$${}_AU_{aa}q^2{}_BU_b^{\,2} + \lambda_{1B}U_{aa}q^2{}_BU_b^{\,2} + {}_AU_{bb}q^2{}_BU_a^{\,2} + \lambda_{1B}U_{bb}q^2{}_BU_a^{\,2} + {}_AU_{c_Ac_A}p^2{}_BU_a^{\,2} + {}_AU_{c_Ac_A}p^2{}_BU_a^{\,2} - {}_AU_{c_Ac_A}p^2{}_BU_{aB}U_b - {}_AU_{ab}2q^2{}_BU_{aB}U_b - \lambda_{1B}U_{ba}2q^2{}_BU_{aB}U_b < 0.$$

The first six terms are negative, the seventh is positive, and the last two are positive when a and b are both substitutes. The more substitutable are a and b, the more likely is it that the second-order conditions will be violated. However, more-than-perfect substitutability will not necessarily cause violation of the second-order conditions (i.e. concavity of preferences). Note, however, that the presence of complementarity strengthens the likelihood of second-order conditions being satisfied.

The evaluation of the second determinant is much more complicated. The result is 26 terms, each of which is a multiple of 6 variables. No unambiguous statement can be made regarding the effect of substitutability or complementarity on the sign of the determinant. As before, more-than-perfect substitutability or complementarity need not violate the second-order conditions.

In summary, the B-K paradox can occur without violating the second-order conditons. It is still, of course, possible that the total conditions are not satisfied, but that is another story.

Conclusions and Extensions

In this paper, we have analyzed the conditions under which oversupply could occur in the presence of external economies. In so doing, we have shown Baumol's analysis to be based upon a misleading definition of "externality". Furthermore, we have shown that the results arrived at by Vincent were faulty for several reasons. For one, the same misleading definitions were used. Furthermore, even if he had used the proper definitions, his results would have been faulty because of the use of improper logic in his partial method. The correct partial method is shown, but even it gives incorrect results. What is required is a simultaneous equation method to determine Δa and Δb, such as is given in this paper.

The above results were derived under some extremely simplifying assumptions, most of which are not crucial to the results. The assumption of constant prices of a and b is merely for simplification. The same holds for the assumption that the second derivatives be a constant. Dropping this would complicate the mathematics considerably. The argument can be further extended to include different goods. That is, a and b need not be the same good "in production". Provided the required substitution relations exist between own consumption and externalities, oversupply is possible in the presence of external economies. The model was developed for two persons, but could be extended to n persons. In this case, one would get n simultaneous equations in n unknowns. As well, the analysis could extend to local governments producing reciprocal spillovers as in Williams (7). Finally, and equally as important, the above equations for Δa and Δb are also appropriate for the case of reciprocal external diseconomies. Here, ${}_AU_b < 0$ and ${}_BU_a < 0$. The same wide range of possibilities avail themselves, including the paradoxical one of undersupply in the presence of external diseconomies.

One need hardly add here that the problem of applying a Pigovian tax-subsidy remedy is not rendered any easier by the above. However, it can still be said that subsidies would be appropriate on each of the individuals to attain the optimum, even though the output might be required to fall, This follows from the first-order conditions.

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